

CFTs of SLEs : the radial case.

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Abstract

We present a relation between conformal field theories (CFT) and radial stochastic Schramm-Loewner evolutions (SLE) similar to that we previously developed for the chordal SLEs. We construct an important local martingale using degenerate representations of the Virasoro algebra. We sketch how to compute derivative exponents and the restriction martingales in this framework. In its CFT formulation, the SLE dual Fokker-Planck operator acts as the two-particle Calogero hamiltonian on boundary primary fields and as the dilatation operator on bulk primary fields localized at the fixed point of the SLE map.

Stochastic Schramm-Loewner evolutions (SLE) [2, 3, 4] are random processes adapted to a probabilistic description of fractal curves or sets growing into simply connected planar domains $\mathbb{U} \subset \mathbb{C}$. They are expected to provide a rigorous description of two dimensional critical clusters in their continuous limit. SLEs depend on a real parameter κ . For $2 \leq \kappa < 4$, they are conjecturally related to the $O(n)$ models in their dilute phase with $n = -2 \cos(4\pi/\kappa)$, and to the Fortuin-Kasteleyn clusters of the Q -state Potts models with $Q = 4 \cos^2(4\pi/\kappa)$ for $4 \leq \kappa < 8$. We refer to refs.[6, 7] for an introduction to SLEs.

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Two classes of SLEs, chordal or radial, have been defined. The former – chordal SLEs – describe random planar curves joining two points on the boundary of a domain \mathbb{U} , while the later – radial SLEs – describe random curves joining a point on the boundary $\partial\mathbb{U}$ to a point in the bulk of \mathbb{U} . The aim of this note is to extend to radial SLEs the relation between SLEs and conformal field theories (CFT), which we developed in the chordal case in ref.[8]. Another relation between CFTs and chordal SLEs has been presented in ref.[10].

In the first section we shall recall the definition of the radial SLEs and its basic covariance properties. The CFT formulation of radial SLEs is given in section 2 in which radial SLEs are viewed as Markov processes in a completion of the enveloping algebra of a Borel subalgebra of the Virasoro algebra. This allows us to define a stochastic evolution operator \mathcal{A} , dual to a Fokker-Planck like operator, which acts on the CFT Hilbert space and whose geometrical meaning is given. As a consequence, we determine in section 3 a key local martingale M_t and we show that the dual Fokker-Planck operator acts as the dilatation operator on spinless conformal primary fields localized in the bulk ending point of the radial SLE curves. This agrees with observations made by Cardy in ref.[11]. The last two sections illustrate the use of the martingale M_t by re-deriving the derivative exponents computed in refs.[4] and by computing the restriction martingales [5] which code for the response of SLEs to deformations of the domains in which they are defined.

1-Radial SLEs. The usual description of radial SLEs is by random curves connecting the point 1 on the boundary of the unit disc to the origin. Its study involves a stochastic differential equation, whose geometric properties play an important role in what follows. If \mathbb{U} is any simply connected domain in \mathbb{C} , x_0 a boundary point of \mathbb{U} and z_* an interior point of \mathbb{U} , there is a unique conformal map from the unit disc to \mathbb{U} mapping 1 to x_0 and 0 to z_* . The image of SLE curves by this map defines a statistical ensemble of random curves in \mathbb{U} , starting at x_0 and ending at z_* , which is by definition radial SLE in (\mathbb{U}, x_0, z_*) . This ensemble is related to a new stochastic differential equation, which we now describe geometrically.

Suppose that $f_t(z)$, $t \in [0, T]$, is a family of functions solving a stochastic differential equation of the form

$$df_t(z) = dt \sigma(f_t(z)) + d\xi_t \rho(f_t(z))$$

with ξ_t a Brownian motion with covariance $\mathbf{E}[\xi_t \xi_s] = \kappa \min(t, s)$. By this we mean that the two functions σ and ρ are holomorphic in some domain

\mathbb{U} and that there is a non empty domain $\mathbb{U}_T \subset \mathbb{U}$ such that f_t maps \mathbb{U}_T into \mathbb{U} and solves the above differential equation for $z \in \mathbb{U}_T$. Suppose that φ maps \mathbb{U} conformally to some domain \mathbb{V} . Then Itô's formula shows that $f_t^\varphi \equiv \varphi \circ f_t \circ \varphi^{-1}$ solves the differential equation $df_t^\varphi = dt \sigma^\varphi \circ f_t^\varphi + d\xi_t \rho^\varphi \circ f_t^\varphi$ with $\rho^\varphi \circ \varphi = \varphi' \rho$ and $\sigma^\varphi \circ \varphi = \varphi' \sigma + \frac{\kappa}{2} \varphi'' \rho^2$. These two relations show that

$$w_{-1} \equiv -\rho(z) \partial_z \quad , \quad w_{-2} \equiv \frac{1}{2} \left(-\sigma(z) + \frac{\kappa}{2} \rho(z) \rho'(z) \right) \partial_z$$

transform as holomorphic vector fields under φ .

When σ and ρ vanish at some point $z_* \in \mathbb{U}$, the equation $df_t = dt \sigma \circ f_t + d\xi_t \rho \circ f_t$ with $f_0(z) = z$ has a unique solution in some nontrivial interval $[0, T]$. It satisfies $f_t(z_*) = z_*$ and $f'_t(z_*) \neq 0$. Inside the space O_{z_*} of germs of holomorphic functions fixing z_* , the subspace $N_{z_*} \equiv \{f \in O_{z_*}, f'(z_*) \neq 0\}$ forms a group for composition, which (anti) acts on O_{z_*} by $\gamma_f \cdot F \equiv F \circ f$. We may view f_t as a random process on N_{z_*} . Another application of Itô's formula shows that $\gamma_{f_t}^{-1} \cdot d\gamma_{f_t} \cdot F = (dt \sigma + d\xi_t \rho) F' + dt \frac{\kappa}{2} \rho^2 F''$, or equivalently

$$\gamma_{f_t}^{-1} \cdot d\gamma_{f_t} = dt \left(-2w_{-2} + \frac{\kappa}{2} w_{-1}^2 \right) - d\xi_t w_{-1}. \quad (1)$$

This equation involves only intrinsic geometric objects.

To define radial SLE on any domain, we only have to choose the vector fields w_{-1} and w_{-2} appropriately: w_{-1} is the generator of conformal motions of \mathbb{U} fixing z_* , and w_{-2} is a holomorphic vector field, unique up to translation by w_{-1} , tangent to $\partial\mathbb{U}$, fixing z_* but with a pole at x_0 .

Notice that $g_t \equiv e^{\xi_t w_{-1}} \cdot f_t$ satisfies $\gamma_{g_t}^{-1} \cdot d\gamma_{g_t} = -2dt (e^{-\xi_t w_{-1}} w_{-2} e^{\xi_t w_{-1}})$, which is an ordinary differential equation.

We observe that the Lie algebra of N_{z_*} is formally isomorphic to a completion of a Borel subalgebra of the Virasoro algebra. In the sequel, we shall make use of the covariance of radial SLE under conformal maps to choose (\mathbb{U}, x_0, z_*) in such a way that w_{-1} and w_{-2} are as simple as possible in terms of Virasoro generators, so that we can make use of its representation theory and of conformal field theory efficiently.

In the outer unit disc geometry $\mathbb{D} = \{z \in \mathbb{C}; |z| \geq 1\}$, $w_{-1} = iz\partial_z$ and $2w_{-2} = z \frac{z+1}{z-1} \partial_z$, and the Loewner equation for the SLE map $g_t = e^{-i\xi_t} f_t$ reads [2]:

$$dg_t(z) = -g_t(z) \frac{g_t(z) + U_t}{g_t(z) - U_t} dt \quad , \quad U_t = e^{i\xi_t}. \quad (2)$$

with $g_0(z) = z$. The SLE hulls \mathbb{K}_t are the sets of points in \mathbb{D} which have been swallowed: $\mathbb{K}_t = \{z \in \mathbb{D}; \tau_z \leq t\}$ with τ_z the swallowing time such that

$g_{\tau_z}(z) = U_{\tau_z}$. The map g_t is the uniformizing map of the complement of \mathbb{K}_t in \mathbb{D} . The SLE curve $\gamma_{[0,\infty)}$, also called the SLE trace, is reconstructed using $g_t(\gamma(t)) = U_t$.

In the upper half plane geometry $\mathbb{H} = \{z \in \mathbb{C}; \Im z \geq 0\}$, $w_{-1} = \frac{1+z^2}{2}\partial_z$ and $2w_{-2} = -\frac{1+z^2}{2z}\partial_z$ with $z_* = i$ and $x_0 = 0$, so that $\tilde{g}_t = (\tilde{f}_t + \eta_t)/(1 - \eta_t \tilde{f}_t)$ satisfies:

$$d\tilde{g}_t(z) = \frac{1 + \tilde{g}_t(z)^2}{2} \left(\frac{1 + \eta_t \tilde{g}_t(z)}{\tilde{g}_t(z) - \eta_t} \right) dt \quad , \quad \eta_t = \tan \xi_t/2.$$

We shall mainly present the results in the case of the disc geometry but they can easily be translated to the upper half plane geometry.

2- The stochastic evolution operator. For making contact with CFT and its operator formalism, it is useful to translate the disc by -1 so that the SLE hulls start to be created at $x_0 = 0$ and grows into \mathbb{D}^{tr} , the complement in \mathbb{C} of the unit disc centered at -1 . We define maps h_t by⁴ $h_t(z) + 1 = U_t^{-1}g_t(z + 1)$. Since we view the hulls as growing outside the unit disc, both maps g_t and h_t fix the point $z_* = \infty$ at infinity: $g_t(z_*) = h_t(z_*) = z_*$. They are normalized there by $g_t(z) = e^{-t}z + O(1)$ and $h_t(z) = e^{-t-i\xi_t}z + O(1)$.

By the results of [9], the maps g_t and h_t are associated to operators G_t and H_t which implement them in CFT. Let L_n be the generators of the Virasoro algebra \mathfrak{vir} with commutation relations: $[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n,-m}$. The operator G_t belongs to the enveloping algebra of a Borel subalgebra of \mathfrak{vir} . From its definition and the radial Loewner equation (2), it follows that G_t satisfies:

$$G_t^{-1} dG_t = L_0 dt + 2 \sum_{n \geq 0} U_t^{n+1} L_{-n-1} dt.$$

The operator H_t is linked to G_t by $H_t = e^{-L_{-1}} G_t e^{i\xi_t L_0} e^{L_{-1}}$. By Itô calculus, one finds that H_t satisfies the stochastic equation:

$$H_t^{-1} dH_t = \left(-2W_{-2} + \frac{\kappa}{2} W_{-1}^2 \right) dt - W_{-1} d\xi_t \quad (3)$$

with $W_{-1} = i(L_0 + L_{-1})$ and $W_{-2} = -\frac{1}{2}(L_0 + 3L_{-1} + 2L_{-2})$. Compare with eq.(1). The stochastic evolution operator \mathcal{A} is by definition the drift term in the stochastic equation, eq.(3):

$$\mathcal{A} \equiv -2W_{-2} + \frac{\kappa}{2} W_{-1}^2. \quad (4)$$

⁴The translation by -1 is for convenience. We could have chosen any other point. The important factor is the dilatation by U_t^{-1} which ensures that the tip $\gamma(t)$ of the SLE trace is mapped at any time to the point x_0 at which it is originally created.

It may be expressed in terms of the Virasoro generators L_n , but those are associated to the vector fields $\ell_n = -z^{n+1}\partial_z$ which are not adapted to the geometry of the disc. Thus, we change basis and consider the generators V_n associated to the vector fields $v_n = -\frac{i^n}{2} \frac{z^{n+1}}{(z+2)^{n-1}} \partial_z$, which are the push forward of the ℓ_n 's by the uniformizing map of \mathbb{D}^{tr} onto the upper half plane \mathbb{H} . The V_n satisfy the Virasoro algebra. The first few are: $V_1 = \frac{i}{2}L_1$, $V_0 = \frac{1}{2}(L_1 + 2L_0)$, $V_{-1} = -\frac{i}{2}(L_1 + 4L_0 + 4L_{-1})$ and $V_{-2} = -\frac{1}{2}(L_1 + 6L_0 + 12L_{-1} + 8L_{-2})$. As a consequence, $W_{-1} = -\frac{1}{2}(V_1 + V_{-1})$ and $W_{-2} = \frac{1}{4}(V_0 + V_{-2})$, and

$$\begin{aligned} \mathcal{A} &= -\frac{1}{2}(V_0 + V_{-2}) + \frac{\kappa}{8}(V_1 + V_{-1})^2 \\ &= \frac{1}{4}(-2V_{-2} + \frac{\kappa}{2}V_{-1}^2) + (\frac{\kappa-2}{4})V_0 + \frac{\kappa}{8}(V_1^2 + 2V_{-1}V_1) \end{aligned} \quad (5)$$

In the upper half plane geometry, the appropriate basis of \mathfrak{vir} are the L_n 's and the stochastic evolution operator is

$$\tilde{\mathcal{A}} = -\frac{1}{2}(L_0 + L_{-2}) + \frac{\kappa}{8}(L_1 + L_{-1})^2$$

The basis V_n 's or L_n 's correspond to two different real forms of \mathfrak{vir} .

We need to recall a few basic facts concerning the Virasoro algebra and its highest weight representations. For the disc geometry we shall consider highest weight vector representations defined with respect to the polarization of the Virasoro algebra associated to the basis V_n . So highest weight vectors $|v\rangle$ are such that $V_n|v\rangle = 0$ for $n > 0$ and $V_0|v\rangle = h|v\rangle$. For the upper half plane geometry, we shall consider highest weight representations with respect to the basis L_n .

We parametrize the conformal weights by

$$h_{r;s} = [(r\kappa - 4s)^2 - (\kappa - 4)^2]/16\kappa$$

for $c = 1 - 6(\kappa - 4)^2/4\kappa$. This may also be written in a coulomb gas representation [12, 13], and we shall need it. We denote by $2\alpha_0$ the background charge so that the central charge is $c = 1 - 12\alpha_0^2$ and the conformal weight of states of coulomb charge α is $h(\alpha) = \frac{1}{2}\alpha(\alpha - 2\alpha_0)$. The weight $h_{r;s}$ corresponds to the charge $\alpha_{r,s} = \alpha_0 - \frac{r}{2}\alpha_+ - \frac{s}{2}\alpha_-$ with α_{\pm} the two screening charges. The correspondance is $\alpha_- = -2\sqrt{2/\kappa}$, $\alpha_+ = \sqrt{\kappa/2}$ and $2\alpha_0 = \alpha_+ + \alpha_-$.

3- Martingales, dilatations and eigenvectors. As in the chordal case [8], a key point is the construction of an important martingale. It is obtained using degenerate representations of the Virasoro algebra with null vectors at level two. We have:

Let $|\omega\rangle$ be the highest weight vector in the irreducible highest weight representation of \mathfrak{vir} of central charge $c = (6 - \kappa)(3\kappa - 8)/2\kappa$ and conformal weight $h_{1;2} = (6 - \kappa)/2\kappa$. Let $2h_{0;1/2} = (6 - \kappa)(\kappa - 2)/8\kappa$. Then,

$$M_t \equiv e^{-2t h_{0;1/2}} H_t |\omega\rangle \quad (6)$$

is a local martingale.

In particular, by projecting this local martingale on vectors $\langle v|$ and assuming appropriate boundedness conditions, we get that the expectations

$$\mathbf{E}[e^{-2t h_{0;1/2}} \langle v| H_t |\omega\rangle]$$

are time independent.

This result follows from the null vector equation $(2V_{-2} - \frac{\kappa}{2}V_{-1}^2)|\omega\rangle = 0$ which selects the representation with conformal weight $h_{1;2}$. As a consequence, $\mathcal{A}|\omega\rangle = 2h_{0;1/2}|\omega\rangle$ with $2h_{0;1/2} = (\frac{\kappa-2}{4})h_{1;2}$ so that $dH_t|\omega\rangle = 2h_{0;1/2}H_t|\omega\rangle dt + H_t W_{-1}|\omega\rangle d\xi_t$.

This result may alternatively be formulated in term of the boundary field $\Psi_{1;2}(x_0)$ creating the state $|\omega\rangle$ at the origin x_0 , the point at which the SLE trace starts. We have:

$$\mathcal{A} \cdot \Psi_{1;2}(x_0) = 2h_{0;1/2} \Psi_{1;2}(x_0). \quad (7)$$

Other properties emerge when testing the evolution operator against conformal primary fields. This amounts to consider correlation functions with insertions of bulk or boundary primary fields $\langle \Phi_{\Delta;\bar{\Delta}}(z, \bar{z}) \cdots \Psi_h(x) \cdots \mathcal{A}|\omega\rangle$. By commuting or deforming contours using standard rules of CFT [1, 14], the action of $\mathcal{A} = -2W_{-2} + \frac{\kappa}{2}W_{-1}^2$ on $|\omega\rangle$ may be traded for an action of

$$\mathcal{A}^T \equiv +2W_{-2} + \frac{\kappa}{2}W_{-1}^2$$

on the bulk and boundary fields.

Recall that for V a Virasoro generator associated to a vector field $v(z)$, bulk primary fields $\Phi_{\Delta;\bar{\Delta}}(z, \bar{z})$ of dimensions $(\Delta, \bar{\Delta})$ and boundary primary fields $\Psi_h(x)$ of dimension h satisfy:

$$\begin{aligned} [V, \Phi_{\Delta;\bar{\Delta}}(z, \bar{z})] &= (v(z)\partial_z + \Delta v'(z) + \bar{v}(\bar{z})\partial_{\bar{z}} + \bar{\Delta}\bar{v}'(\bar{z})) \Phi_{\Delta;\bar{\Delta}}(z, \bar{z}) \\ [V, \Psi_h(x)] &= (v(x)\partial_x + h \Re v'(x)) \Psi_h(x) \end{aligned}$$

Under global conformal maps $h_t(z)$ implemented in CFT by operators H_t , primary fields transform as:

$$\begin{aligned} H_t^{-1} \Phi_{\Delta;\bar{\Delta}}(z, \bar{z}) H_t &= h'_t(z)^\Delta \bar{h}'_t(\bar{z})^{\bar{\Delta}} \Phi_{\Delta;\bar{\Delta}}(h_t(z), \bar{h}_t(\bar{z})) \\ H_t^{-1} \Psi_h(x) H_t &= |h'_t(x)|^h \Psi_h(h_t(x)) \end{aligned}$$

As a consequence⁵, the action of \mathcal{A}^T on primary fields is easy to compute using the explicit expressions of the vector fields associated to W_{-2} and W_{-1} .

For bulk primary fields localized at the point z_* fixed by the SLE map, we get the particularly nice result:

$$\mathcal{A}^T \cdot \Phi_{\Delta, \bar{\Delta}}(z_*, \bar{z}_*) = \left(d - \frac{\kappa}{2} s^2\right) \Phi_{\Delta, \bar{\Delta}}(z_*, \bar{z}_*) \quad (8)$$

with $d = \Delta + \bar{\Delta}$ the scaling dimension and $s = \Delta - \bar{\Delta}$ the spin. In other words, \mathcal{A}^T , which may be thought of as a dual of a Fokker-Planck operator, acts diagonally on primary operators localized at the fixed point. For spinless operators, this action is simply the dilatation. This is in agreement with the points raised in ref.[11].

On boundary conformal fields, \mathcal{A}^T acts as a second order differential operator \mathcal{H}_h closely related to the Calogero hamiltonian. In the case of the translated disc geometry, with $x = e^{i\theta} - 1$ parametrizing the boundary of \mathbb{D}^{tr} , this action reads:

$$\mathcal{A}^T \cdot \Psi_h(x) = \mathcal{H}_h \cdot \Psi_h(x) \equiv \left(\frac{\kappa}{2} \partial_\theta^2 + \cotan \frac{\theta}{2} \partial_\theta - \frac{h}{2 \sin^2 \theta/2} \right) \Psi_h(x) \quad (9)$$

The three properties (7,8,9) have a simple consequence: appropriate CFT correlation functions are eigenfunctions of \mathcal{H}_h . For instance,

$$(\mathcal{H}_h - \epsilon_{\Delta, \bar{\Delta}}) \cdot \langle \Phi_{\Delta, \bar{\Delta}}(z_*, \bar{z}_*) \Psi_h(x) \Psi_{1;2}(x_0) \rangle = 0 \quad (10)$$

with \mathcal{H}_h defined above, eq.(9), and eigenvalue

$$\epsilon_{\Delta, \bar{\Delta}} = 2h_{0;1/2} - d + \frac{\kappa}{2} s^2 \quad (11)$$

with $d = \Delta + \bar{\Delta}$ and $s = \Delta - \bar{\Delta}$.

The simplest case is for Ψ_h the identity operator. The non vanishing of $\langle \Phi_{\Delta, \bar{\Delta}}(z_*, \bar{z}_*) \Psi_{1;2}(x_0) \rangle$ then requires $\epsilon_{\Delta, \bar{\Delta}} = 0$, or equivalently $d = 2h_{0;1/2} + \frac{\kappa}{2} s^2$, which is indeed the fusion rule relation. This case also provides a simple check of the martingale $M_t = e^{-2t h_{0;1/2}} H_t | \omega \rangle$. Indeed, we may compute $e^{-2t h_{0;1/2}} \langle \Phi_{\Delta, \bar{\Delta}}(z_*, \bar{z}_*) H_t | \omega \rangle$ by moving H_t to the left so that $\Phi_{\Delta, \bar{\Delta}}$ gets transformed by $h_t(z)$. Using the normalisation of the SLE map at the fixed point z_* and the fusion rule $d = 2h_{0;1/2} + \frac{\kappa}{2} s^2$, we get $\exp[is\xi_t + \frac{\kappa}{2} s^2 t]$ which is a well known martingale for the Brownian motion.

Some of the previous results have a simple interpretation in terms of the $O(n)$ models in the dilute phase. Using Coulomb gas techniques, the L -leg

⁵Note that one has to be careful with the real form involved in the definition of the generator V .

bulk operators have been identified [12, 15] with the bulk operator $\Phi_{0;L/2}$, and the L -leg boundary operators with $\Psi_{1;L+1}$. Hence the boundary operator $\Psi_{1;2}$ singled out by the SLE martingale M_t corresponds to a 1-leg operator creating a single curve, while the bulk operator $\Phi_{0;1/2}$ to which it couples corresponds to the termination of a single curve in the bulk as it should be.

4- Derivative exponents. Assume $\kappa > 4$. Derivative exponents code for the asymptotic behavior of expectations $f_h(x, t) \equiv \mathbf{E}[|h'_t(x)|^h \mathbf{1}_{\{\tau_x > t\}}]$, $h \geq 0$, at large time for x on the boundary. In particular $f_0(x, t)$ is the probability that the point x has not been swallowed by the SLE trace up to time t . As shown in [4] using probabilistic arguments, the time evolution of $f_h(x, t)$ is governed by \mathcal{H}_h , eq.(9): $\partial_t f_h(x, t) = \mathcal{H}_h \cdot f_h(x, t)$. So its large time behavior is dictated by the eigenstate of \mathcal{H}_h of largest eigenvalue. We shall identify this eigenvalue using the martingale $M_t = e^{-2t h_{0;1/2}} H_t |\omega\rangle$.

Indeed, consider as above the projection of the martingale M_t on the state created by primary fields localized at the fixed point and on the boundary:

$$F_h(x, t) \equiv \langle \Phi_{\Delta, \Delta}(z_*, \bar{z}_*) \Psi_h(x) H_t |\omega\rangle$$

By construction $e^{-2t h_{0;1/2}} F_h(x, t)$, $h \geq 0$, is a local martingale. It may be computed by moving H_t to the left, which conformally transforms the primary fields. Hence, $F_h(x, t) = e^{2\Delta t} |h'_t(x)|^h F_h(h_t(x), 0)$ where we used the known asymptotic behavior of h_t at the fixed point z_* . As a consequence:

$$\mathbf{E}[|h'_t(x)|^h F_h(h_t(x), 0)] = e^{\epsilon_{\Delta, \Delta} t} F_h(x, 0)$$

with $\epsilon_{\Delta, \Delta} = 2h_{0;1/2} - 2\Delta$. This is of course related to the eigen-equation (10).

Now, as a consequence of the null vector relation $(2V_{-2} - \frac{\kappa}{2} V_{-1}^2) |\omega\rangle = 0$, the function $F_h(x, 0) = \langle \Phi_{\Delta, \Delta}(z_*, \bar{z}_*) \Psi_h(x) \Psi_{1;2}(x_0) \rangle$ satisfies a second order differential equation [1] which depends on Δ . The primary field $\Phi_{\Delta, \Delta}(z_*, \bar{z}_*)$ is chosen by demanding that $F_h(x, 0)$ satisfies the same boundary condition as $f_h(x, t)$. Namely [4], it is single valued when x moves along the boundary and it vanishes when x approaches x_0 from both sides. This selects the conformal weight $\Delta(h)$,

$$2\Delta(h) = \frac{h}{2} + 2h_{0;1/2} + \frac{\kappa}{8} \delta_+(h)$$

with $\delta_{\pm}(h) = \left[\kappa - 4 \pm \sqrt{(\kappa - 4)^2 + 16h\kappa} \right] / 2\kappa$. With this choice, $F_h(x, 0) = [\sin \theta/2]^{\delta_+(h)}$ in the disc geometry. This function has no node, it is thus the

fundamental. As a consequence, $f_h(x, t)$ decreases exponentially as $e^{-\lambda(h)t}$ with an exponent:

$$\lambda(h) = 2\Delta(h) - 2h_{0;1/2} = \frac{h}{2} + \frac{1}{16} \left[\kappa - 4 + \sqrt{(\kappa - 4)^2 + 16h\kappa} \right]$$

It of course agrees with ref.[4] and with the computations of ref.[16], section 12.3, based on 2D quantum gravity.

The dimension $\Delta(h)$ has a simple interpretation in the Coulomb gas representation. Let $\beta_\kappa = \sqrt{2/\kappa}$ be the charge of $\Psi_{1;2}$ creating the SLE trace and β , or $2\alpha_0 - \beta$, be the charge of the boundary operator Ψ_h with $h = \frac{1}{2}\beta(\beta - 2\alpha_0) \geq 0$. Then $\delta_+(h) = \beta_\kappa\beta$ and $\delta_-(h) = \beta_\kappa(2\alpha_0 - \beta)$ with $\beta > \alpha_0 > 0$, reflecting the fact that the fusion relations with $\Psi_{1;2}$ are linear in terms of Coulomb charges. $\delta_\pm(h)$ are directly related to the dimensions of the operators produced by fusing Ψ_h with $\Psi_{1;2}$ since the operator product expansion $\Psi_h(x)\Psi_{1;2}(x_0)$ behaves as $(x - x_0)^{\delta_+(h)}$ or as $(x - x_0)^{\delta_-(h)}$ for $x \rightarrow x_0$. Hence, the vanishing of $\langle \Phi_{\Delta,\Delta}(z_*, \bar{z}_*)\Psi_h(x)\Psi_{1;2}(x_0) \rangle$ as $x \rightarrow x_0$ demands to represent Ψ_h with the charge β , $\beta > \alpha_0$ and not with $2\alpha_0 - \beta$. Let $\alpha = \bar{\alpha}$ be the charges of the bulk operator $\Phi_{\Delta,\Delta}$. Demanding that there are no screening charges in the Coulomb gas representation of the correlation function $\langle \Phi_{\Delta,\Delta}(z_*, \bar{z}_*)\Psi_h(x)\Psi_{1;2}(x_0) \rangle$ ensures that this function has no monodromy, as we required. Since one has to compensate the background charge $2\alpha_0$, the absence of screening charges imposes $2\alpha + \beta + \beta_\kappa = 2\alpha_0$ or $2\alpha + (2\alpha_0 - \beta) + \beta_\kappa = 2\alpha_0$. Demanding now that this correlation vanishes as x approaches x_0 selects the charge $2\alpha = 2\alpha_0 - \beta - \beta_\kappa$. The corresponding scaling dimension $\alpha(\alpha - 2\alpha_0)$ is $2\Delta(h)$.

5- The restriction martingales. The martingale M_t may be used to construct the restriction martingales [5] coding for the influence of deformations of domains on radial SLEs. For simplicity we present it in the case of the disc geometry. The construction is similar to the one we presented in [9], so we shall only sketch it. Let A be a hull in \mathbb{D}^{tr} and ϕ_A be one of the uniformizing map of its complement onto \mathbb{D}^{tr} fixing x_0 . Given ϕ_A and h_t , we may write in a unique way a commutative diagram $\phi_{\hat{A}_t} \circ h_t = \hat{h}_t \circ \phi_A$ where $\phi_{\hat{A}_t}$ (resp. \hat{h}_t) uniformizes the complement of $h_t(A)$ (resp. $\phi_A(\mathbb{K}_t)$) onto \mathbb{D}^{tr} with $\phi_{\hat{A}_t}$ fixing x_0 and \hat{h}_t fixing $z_* = \infty$. Let as above H_t (resp. \hat{H}_t) be the operators implementing h_t (resp. \hat{h}_t) in CFT. Similarly, let G_A (resp. \hat{G}_{A_t}) be those implementing ϕ_A (resp. $\phi_{\hat{A}_t}$). Then [9],

$$G_A^{-1} H_t = Z_t(A) \hat{H}_t \hat{G}_{A_t}^{-1}$$

with

$$Z_t(A) = \exp \frac{c}{6} \int_0^t ds (S\phi_{\hat{A}_s})(x_0).$$

with $(S\phi)$ the Schwarzian derivative of ϕ .

By construction $e^{-2t h_{0;1/2}} G_A^{-1} H_t |\omega\rangle$ is a local martingale. We may project it on the bulk conformal operator $\Phi_{0;1/2}$ of dimension $2h_{0;1/2}$ located at the fixed point $z_* = \infty$. Computing $\langle \Phi_{0;1/2}(z_*, \bar{z}_*) G_A^{-1} H_t |\omega\rangle$ using the commutative diagram yields the martingale ⁶ :

$$M_t(A) \equiv e^{-2t h_{0;1/2}} |\hat{h}'_t(z_*)|^{-2h_{0;1/2}} |\phi'_{\hat{A}_t}(x_0)|^{h_{1;2}} Z_t(A)$$

Alternatively, since $h'_t(z_*) \phi'_{\hat{A}_t}(z_*) = \hat{h}'_t(z_*) \phi'_A(z_*)$ and $|h'_t(z_*)| = e^{-t}$, this reads:

$$M_t(A) |\phi'_A(z_*)|^{-2h_{0;1/2}} = |\phi'_{\hat{A}_t}(x_0)|^{h_{1;2}} |\phi'_{\hat{A}_t}(z_*)|^{-2h_{0;1/2}} Z_t(A)$$

As in [5], this formula may be used to evaluate the probability that the radial SLE hull at $\kappa = 8/3$ does not touch the hull A . This martingale may be further generalized by projecting $G_A^{-1} H_t |\omega\rangle$ on bulk operators with spin $\Phi_{\Delta, \bar{\Delta}}(z_*, \bar{z}_*)$ satisfying the fusion rule $d = 2h_{0;1/2} + \frac{\kappa}{2} s^2$.

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⁶Recall that since $z_* = \infty$, the local coordinate around z_* is $1/z$.

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